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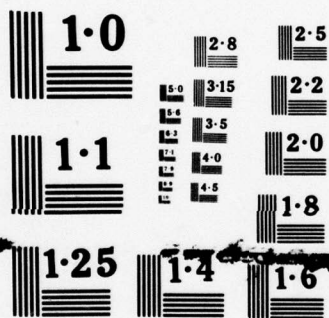
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IN CERTAIN RANDOM WALK PROBLEMS,

BY

10 D. SIEGMUND

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## CORRECTED DIFFUSION APPROXIMATIONS IN CERTAIN RANDOM WALK PROBLEMS

by

D. Siegmund

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### Abstract

Correction terms are obtained for the diffusion approximation to one and two barrier ruin problems in finite and infinite time. The corrections involve moments of ladder height distributions, and a method is given for calculating them numerically. Examples show that the corrected approximations can be much more accurate than the originals.

Key Words and Phrases: Diffusion approximation, heavy traffic, random walk, gambler's ruin.

# CORRECTED DIFFUSION APPROXIMATIONS IN CERTAIN RANDOM WALK PROBLEMS

## 1. Introduction and Summary

Let  $x_1, x_2, \dots$ , be independent and identically distributed with mean  $\mu = E(x_1)$ . Let  $s_n = x_1 + \dots + x_n$ , and for  $a \leq 0 < b$  define the stopping times

$$\tau = \tau(b) = \inf\{n : s_n > b\} \quad (\tau_+ = \tau(0))$$

and

$$T = T(a, b) = \inf\{n : s_n \notin [a, b]\}.$$

The probabilities

$$(1) \quad P\{\tau < \infty\}$$

and

$$(2) \quad P\{s_T > b\}$$

as well as their "finite time" analogues

$$(3) \quad P\{\tau \leq m\}$$

and

$$(4) \quad P\{T \leq m, s_T > b\}$$

arise in a wide variety of contexts including insurance risk theory, queueing theory, storage theory, and sequential statistical analysis.

One very simple and useful approximation to these probabilities is the so-called diffusion approximation (in queueing theory--a

"heavy traffic" approximation), in which the discrete random walk is replaced by an appropriate Brownian motion process, for which (1)-(4) may be computed exactly. A defect of this approximation is that in many cases it is not sufficiently accurate; and a variety of methods, both theoretical and numerical have been suggested as alternatives in various special cases.

One alternative which deserves particular attention is the beautiful result of Cramér as extended by Feller (see Feller, 1966, pp. 363 and 393) that as  $b \rightarrow \infty$

$$(5) \quad P\{\tau < \infty\} \sim Ce^{-\gamma b}.$$

Here  $f(\lambda) = E \exp(\lambda s_1)$  is assumed finite and  $\gamma$  is the positive value (assumed to exist) for which  $f(\gamma) = 1$ . One difficulty in using (5) is determining  $C$ , which in general is a fairly complicated functional of the distribution of  $s_{\tau_+}$ . Often it can be computed numerically. (See Woodroffe, 1978, for the first general results in this direction.) This approach seems to be less successful in dealing with (2), (3), and (4), although some results have been obtained by Borovkov (1962), von Bahr (1974), and Siegmund (1975a,b).

The purpose of this paper is to suggest usable approximations to (1)-(4) and related quantities by computing correction terms in the diffusion approximation. These correction terms involve moments of  $s_{\tau_+}$ , and a numerical method for computing them is described.

The paper is arranged as follows. Section 2 discusses the simpler problem of approximating  $E(\sup_n s_n) = \int_0^\infty P\{\tau(b) < \infty\} db$ , which, however, already involves the most novel mathematical results

(see especially Lemma 4). Section 3 is devoted to (1) and (2) and Section 4 to (3) and (4). The treatment in Section 4 uses Laplace transforms. Although formal inversion of the transforms is easy and yields what appear to be useful approximations, a rigorous justification of the inversions remains an open problem. Section 5 discusses numerical evaluation of the constants entering into the corrected approximation. Section 6 contains numerical examples. The practically oriented reader may wish to read these sections in the reverse order.

A result closely related to Theorem 2 of Section 3 was obtained by completely different methods by Borovkov (1965).

The following notation and assumptions are used throughout. Assume that for  $\theta$  in some open interval containing 0, under  $P_\theta$ ,  $x_1, x_2, \dots$ , are independent random variables with probability density function

$$(6) \quad \exp\{\theta x - \psi(\theta)\}$$

relative to some non-arithmetic measure  $F$ . (For arithmetic  $F$  analogous but slightly different results are obtained.) The function  $\psi$  is normalized so that  $\psi(0) = \psi'(0) = 0$ ,  $\psi''(0) = 1$ , and then  $F$  is the  $P_0$  distribution of  $x_k$ . It is easily verified that

$$\psi'(\theta) = E_\theta(x_1), \quad \psi''(\theta) = \text{var}_\theta(x_1) \quad ,$$

and

$$E_\theta x_1 < , = , \text{ or } > 0 \quad \text{according as} \quad \theta < , = , \text{ or } > 0 \quad .$$

The function  $\psi$  is convex and hence to small  $\theta \neq 0$  there corresponds exactly one  $\tilde{\theta} \neq 0$ , necessarily of opposite sign, for which  $\psi(\theta) = \psi(\tilde{\theta})$ . It will be convenient to think of  $\theta_0 < 0$  as given and  $\theta_1 > 0$  defined by

$$(7) \quad \psi(\theta_0) = \psi(\theta_1) \quad ,$$

although this relation might well be reversed.

Let  $P_\theta^{(n)}$  be the restriction of  $P_\theta$  to the space of  $x_1, \dots, x_n$ , so that by (6), for all  $\theta'$  and  $\theta''$

$$(8) \quad dP_{\theta'}^{(n)} = \exp\{(\theta' - \theta'')s_n - n[\psi(\theta') - \psi(\theta'')]\}dP_{\theta''}^{(n)} \quad .$$

In particular, by (7)

$$dP_{\theta_1}^{(n)} = \exp\{(\theta_1 - \theta_0)s_n\}dP_{\theta_0}^{(n)} \quad .$$

The following version of the fundamental identity of sequential analysis will be used repeatedly.

Lemma 1. Let  $\sigma$  be a stopping time and  $f$  a non-negative random variable such that for all  $n = 1, 2, \dots$ ,  $fI_{\{\sigma=n\}}$  is a function of  $x_1, \dots, x_n$  (and not  $x_{n+1}, \dots$ ). Then for any pair  $\theta', \theta''$

$$\int_{\{\sigma < \infty\}} f dP_{\theta'} = \int_{\{\sigma < \infty\}} f \exp\{(\theta' - \theta'')s_\sigma - \sigma[\psi(\theta') - \psi(\theta'')]\} dP_{\theta''} \quad .$$

The proof follows at once by writing  $\int_{\{\sigma < \infty\}} = \sum_{n=1}^{\infty} \int_{\{\sigma=n\}}$  and using (8).

$$2. E_{\theta_0}(\sup_{n \geq 0} s_n)$$

The following result is elegant and illustrates the basic mathematical techniques of this paper. The proof of Lemma 4 seems particularly interesting. Recall that  $\tau_+ = \inf\{n : s_n > 0\}$ . It will be convenient to use the notation  $\Delta = \theta_1 - \theta_0$  and  $W = \sup_{n \geq 0} s_n$ . Observe that as  $\theta_0 \uparrow 0$ ,  $\theta_1 \downarrow 0$  and hence  $\Delta \rightarrow 0$ .

Theorem 1. As  $\theta_0 \uparrow 0$

$$(9) \quad E_{\theta_0} W = \Delta^{-1} - \frac{E_0 s_{\tau_+}^2}{2E_0 s_{\tau_+}} + \frac{\Delta}{2} \left[ \frac{1}{3} \frac{E_0 s_{\tau_+}^3}{E_0 s_{\tau_+}} - \left( \frac{E_0 s_{\tau_+}^2}{2E_0 s_{\tau_+}} \right)^2 \right] + o(\Delta)$$

and

$$(10) \quad E_{\theta_0} W^2 = 2\Delta^{-2} - \Delta^{-1} \frac{E_0 s_{\tau_+}^2}{E_0 s_{\tau_+}} + \left( \frac{E_0 s_{\tau_+}^2}{2E_0 s_{\tau_+}} \right)^2 + o(1) .$$

Proof. Let  $\tau_+^{(n)}$  denote the  $n^{\text{th}}$  increasing ladder epoch ( $\tau_+^{(1)} = \tau_+$ ).

Let  $p = P_{\theta_0} \{\tau_+ < \infty\}$  and  $\xi = E_{\theta_0}(s_{\tau_+} | \tau_+ < \infty)$ . Then from

$$\begin{aligned} P_{\theta_0} \{W > x\} &= \sum_{n=1}^{\infty} P_{\theta_0} \{\tau_+^{(n)} < \infty, s_{\tau_+^{(n)}} > x, \tau_+^{(n+1)} = \infty\} \\ &= (1-p) \sum_{n=1}^{\infty} P_{\theta_0} \{\tau_+^{(n)} < \infty, s_{\tau_+^{(n)}} > x\} \end{aligned}$$

follows

$$\begin{aligned}
E_{\theta_0} W &= (1-p) \sum_1^{\infty} p^n E_{\theta_0} (s_{\tau_+}^{(n)} | \tau_+^{(n)} < \infty) = (1-p) \sum_1^{\infty} np^n \xi = p\xi/(1-p) . \\
&= \int_{\{\tau_+ < \infty\}} s_{\tau_+} dP_{\theta_0} / P_{\theta_0} \{\tau_+ = \infty\} .
\end{aligned}$$

Hence by Lemma 1

$$(11) \quad E_{\theta_0} W = \frac{E_{\theta_1} [s_{\tau_+} \exp(-\Delta s_{\tau_+})]}{1 - E_{\theta_1} \exp(-\Delta s_{\tau_+})} .$$

It follows from Taylor series expansions that

$$E_{\theta_1} [s_{\tau_+} \exp(-\Delta s_{\tau_+})] = E_{\theta_1} s_{\tau_+} - \Delta E_{\theta_1} s_{\tau_+}^2 + \frac{1}{2} \Delta^2 E_{\theta_1} s_{\tau_+}^3 + o(\Delta^3)$$

and

$$1 - E_{\theta_1} \exp(-\Delta s_{\tau_+}) = \Delta E_{\theta_1} s_{\tau_+} - \frac{1}{2} \Delta^2 E_{\theta_1} s_{\tau_+}^2 + \frac{1}{6} \Delta^3 E_{\theta_1} s_{\tau_+}^3 + o(\Delta^4).$$

Substituting these expressions into (11) and expanding yields

$$(12) \quad E_{\theta_1} W = \Delta^{-1} - \frac{1}{2} \frac{E_{\theta_1} s_{\tau_+}^2}{E_{\theta_1} s_{\tau_+}} + \frac{\Delta E_{\theta_1} s_{\tau_+}^3}{3 E_{\theta_1} s_{\tau_+}} - \Delta \left( \frac{E_{\theta_1} s_{\tau_+}^2}{2 E_{\theta_1} s_{\tau_+}} \right)^2 + o(\Delta^2) .$$

By Lemma 2 below

$$(13) \quad E_{\theta_1} s_{\tau_+}^k = E_{\theta_0} s_{\tau_+}^k + \theta_1^k E_{\theta_0} s_{\tau_+}^{k+1} / (k+1) + o(\Delta) .$$

Observing that  $\Delta \sim 2\theta_1$  and substituting (13) into (12) yields (9).

The proof of (10) is similar and has been omitted.

Lemma 2. For each  $k > 0$ , as  $\theta_1 \downarrow 0$

$$E_{\theta_1} s_{\tau_+}^k = E_0 s_{\tau_+}^k + \theta_1^k E_0 s_{\tau_+}^{k+1} / (k+1) + o(\theta_1) .$$

Proof. From the representation

$$E_{\theta} s_{\tau_+}^k = E_0 \{ s_{\tau_+}^k \exp [\theta s_{\tau_+} - \tau_+ \psi(\theta)] \}$$

and the dominated convergence theorem it is easy to see that

$$(14) \quad \lim_{\theta \downarrow 0} E_{\theta} s_{\tau_+}^k = E_0 s_{\tau_+}^k .$$

This representation also shows that  $f(\theta) = E_{\theta} s_{\tau_+}^k$  is continually differentiable on  $(0, \varepsilon)$  for some  $\varepsilon > 0$  and

$$(15) \quad f'(\theta) = E_{\theta} [s_{\tau_+}^k (s_{\tau_+} - \mu_{\tau_+})] ,$$

where  $\mu_{\theta} = \psi'(\theta) = E_{\theta} x_1$ . It will be shown in Lemma 4 below that

$$(16) \quad \lim_{\theta \downarrow 0} \mu_{\theta} E_{\theta} \tau_+ s_{\tau_+}^k = E_0 s_{\tau_+}^{k+1} / (k+1) .$$

Writing

$$f(\theta_1) = f(\theta) + (\theta_1 - \theta) f'(\theta) + \int_{\theta}^{\theta_1} [f'(y) - f'(\theta)] dy ,$$

letting  $\theta \rightarrow 0$ , and appealing to (14)-(16) completes the proof.

Let  $\phi(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2} x^2)$  and  $\Phi(x) = \int_{-\infty}^x \phi(y) dy$ . For  $t > 0$ ,  $-\infty < \xi < \infty$ , and  $c > 0$  let

$$G(t; \xi, c) = 1 - \Phi(ct^{-1/2} - \xi t^{1/2}) + e^{2\xi c} \Phi(-ct^{-1/2} - \xi t^{1/2}) .$$

Let  $H(x) = (E_0 s_{\tau_+})^{-1} \int_0^x P_0\{s_{\tau_+} > y\} dy$ . It follows from the invariance principle of Erdős and Kac (1946) that if  $b \rightarrow \infty$  and  $\theta \rightarrow 0$  such that  $\theta b \rightarrow \xi \in (-\infty, \infty)$ , then

$$(17) \quad \lim_{\theta} P_{\theta}\{\tau \leq b^2 t\} = G(t; \xi, 1) \quad (0 < t < \infty),$$

or equivalently

$$(18) \quad \lim_{\theta} E_{\theta} \exp(-\lambda \tau / b^2) = \exp\{-(2\lambda + \xi^2)^{1/2} - \xi\}$$

uniformly in  $\lambda \geq 0$ . From consideration of the renewal process defined by the increasing ladder heights and the renewal theorem (e.g., Feller, 1966, p. 354), it follows that

$$(19) \quad \lim_{b \rightarrow \infty} P_0\{s_{\tau} - b \leq x\} = H(x) \quad (0 < x < \infty).$$

The next lemma extends (19) to the case  $\theta b \rightarrow \xi \geq 0$  and establishes the asymptotic independence of  $s_{\tau} - b$  and  $\tau/b^2$ . For the case  $\theta = 0$  it was stated by Siegmund (1975b). The method of proof is a modification of the argument of that paper.

Lemma 3. Let  $\theta \geq 0$ . Suppose  $b \rightarrow \infty$  and  $\theta \rightarrow 0$  in such a way that  $\theta b \rightarrow \xi \in [0, \infty)$ . Then

$$(20) \quad \lim_{\theta} P_{\theta}\{\tau \leq b^2 t, s_{\tau} - b \leq x\} = G(t; \xi, 1) H(x).$$

Proof. Let  $m = [b^2 t]$  ( $[\cdot]$  = integer part).

$$(21) \quad P_{\theta}\{\tau(b) > m, s_{\tau(b)} - b \leq x\} \\ = \int_{[0, \infty)} P_{\theta}\{\tau(b) > m, s_m \in b - dy\} P_{\theta}\{s_{\tau(y)} - y \leq x\}.$$

Consider splitting the integral in (21) into three pieces according as  $0 \leq y < \varepsilon b$ ,  $\varepsilon b \leq y \leq b/\varepsilon$ , or  $b/\varepsilon < y < \infty$ , where  $0 < \varepsilon < 1$  is given. It follows easily from the central limit theorem that as  $b \rightarrow \infty$

$$\overline{\lim} P_{\theta} \{ \tau(b) > m, b(1-\varepsilon) < s_m \leq b \} \leq \overline{\lim} P_{\theta} \{ b(1-\varepsilon) < s_m \leq b \} \leq \text{const. } \varepsilon,$$

so the first integral is small provided  $\varepsilon$  is. Similarly, the integral over  $b/\varepsilon < y < \infty$  may be shown to be small for sufficiently small  $\varepsilon$  and all large  $b$ . Consider first the case  $\theta = 0$ . Then by (19), over the range  $\varepsilon b \leq y \leq b/\varepsilon$  the integrand in (21) converges uniformly to  $H(x)$ , which together with (17) completes the proof in this special case.

By Lemma 1

$$(22) \quad P_{\theta} \{ s_{\tau(y)} - y \leq x \} = e^{\theta y} \int_{\{s_{\tau(y)} - y \leq x\}} \exp\{\theta [s_{\tau(y)} - y] - \tau(y)\psi(\theta)\} dP_0.$$

Also,  $\psi(\theta) \sim \theta^2/2$  ( $\theta \rightarrow 0$ ). Hence, by (22), (18) with  $\theta = 0$ , and the special case of (20) with  $\theta = 0$  one obtains

$$\lim P_{\theta} \{ s_{\tau(y)} - y \leq x \} = H(x)$$

uniformly in  $bc \leq y \leq bc^{-1}$ . Repeating the first part of the argument with this strengthened form of (19) completes the proof in the general case.

Lemma 4.  $\lim_{\theta \rightarrow 0} \mu_{\theta} E_{\theta} \tau_+ s_{\tau_+}^k = E_0 s_{\tau_+}^{k+1} / (k+1) \quad (k > 0).$

Proof. For  $\theta > 0$

$$\mu E_{\theta} (\tau_+ s_{\tau_+}^k) = \mu \sum_{n=0}^{\infty} \int_{\{\tau_+ > n\}} s_{\tau_+}^k dP_{\theta} = \mu \sum_{n=0}^{\infty} \int_{[0, \infty)} E_{\theta} [s_{\tau(x)} - x]^k P_{\theta} \{ \tau_+ > n, s_n \in -dx \}.$$

Let  $M = \min_{0 \leq n < \infty} s_n$ , and let  $\tau_-^{(r)}$  denote the  $r^{\text{th}}$  weak descending ladder epoch. Then

$$\begin{aligned}
 \sum_{n=0}^{\infty} P_{\theta} \{ \tau_+ > n, s_n < -x \} &= \sum_{n=0}^{\infty} P_{\theta} \{ s_n - s_k \leq 0 \forall 0 \leq k \leq n, s_n < -x \} \\
 &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} P_{\theta} \{ \tau_-^{(r)} = n, s_n < -x \} \\
 &= E_{\theta} \tau_+ \sum_{r=0}^{\infty} P_{\theta} \{ \tau_-^{(r)} < \infty, s_{\tau_-^{(r)}} < -x \} P_{\theta} \{ \tau_- = \infty \} \\
 &= E_{\theta} \tau_+ \sum_{r=0}^{\infty} P_{\theta} \{ \tau_-^{(r)} < \infty, s_{\tau_-^{(r)}} < x, \tau_-^{(r+1)} = \infty \} \\
 &= E_{\theta} \tau_+ P_{\theta} \{ M < -x \} .
 \end{aligned}$$

Here the third equality follows from the well-known identity

$E_{\theta} \tau_+ = 1/P_{\theta} \{ \tau_- = \infty \}$  (cf. Feller, 1966, p. 379). Hence by Wald's identity and Lemma 1

$$\begin{aligned}
 (23) \quad \mu E_{\theta} (\tau_+ s_{\tau_+}^k) &= E_{\theta} s_{\tau_+} \int_{[0, \infty)} E_{\theta} [s_{\tau(x)} - x]^k P_{\theta} \{ M \in -dx \} \\
 &= E_{\theta} s_{\tau_+} \int_{[0, \infty)} E_0 [s_{\tau(x/\theta)} - x/\theta]^k \\
 &\quad \exp [\theta (s_{\tau(x/\theta)} - x/\theta) - \psi(\theta) \tau(x/\theta)] e^x P_{\theta} \{ M \in -dx/\theta \} .
 \end{aligned}$$

By the renewal theorem, (18) with  $\theta = 0$ , and Lemma 3

$$\begin{aligned}
(24) \quad & \lim_{\theta \downarrow 0} E_0 \{ [s_{\tau(x/\theta)} - x/\theta]^k \exp[\theta(s_{\tau(x/\theta)} - x/\theta) - \psi(\theta)\tau(x/\theta)] \} \\
& = e^{-x} E_0 s_{\tau_+}^{k+1} / (k+1) E_0 s_{\tau_+} .
\end{aligned}$$

Moreover, an examination of the proof of (24) shows that the indicated convergence holds uniformly in  $[\epsilon, \epsilon^{-1}]$  for each  $\epsilon > 0$ , and the expectation on the left is bounded uniformly in  $x$  and (small)  $\theta$ . An easy argument based on Lemma 1 (cf. Section 3, especially (25)) shows that

$$P_{\theta} \{M \leq -x/\theta\} \rightarrow e^{-2x} \quad (x \geq 0) .$$

Also for sufficiently small  $\theta$

$$P_{\theta} \{M \leq -x/\theta\} \leq \exp(-3x/2) .$$

Hence, partitioning the range of integration in (23) into  $[0, \epsilon]$ ,  $[\epsilon, \epsilon^{-1}]$ , and  $(\epsilon^{-1}, \infty)$ , and substituting (24) yields

$$\lim_{\theta \downarrow 0} \mu E_{\theta}(\tau_+ s_{\tau_+}^k) = (k+1)^{-1} E_0 s_{\tau_+}^{k+1} \int_{[0, \infty)} 2e^{-2x} dx ,$$

which completes the proof.

$$3. \quad P_{\theta_0} \{\tau < \infty\} \text{ and } P_{\theta_0} \{s_{\tau} > b\}$$

Suppose that  $b \rightarrow \infty$  and  $\theta_0 \uparrow 0$  in such a way that  $b\theta_0 \rightarrow -\xi < 0$ . Then  $b\theta_1 \rightarrow \xi$  and  $b\Delta = b(\theta_1 - \theta_0) \rightarrow 2\xi$ . From Lemma 1 and a Taylor expansion it follows that

$$\begin{aligned}
(25) \quad P_{\theta_0} \{\tau < \infty\} &= e^{-\Delta b} E_{\theta_1} \exp[-\Delta(s_\tau - b)] \\
&= e^{-\Delta b} [1 - \Delta E_{\theta_1}(s_\tau - b) + \frac{1}{2} \Delta^2 E_{\theta_1}(s_\tau - b)^2 + o(\Delta^3)] .
\end{aligned}$$

By Lemma 1

$$E_{\theta_1}(s_\tau - b) = E_0\{(s_\tau - b) \exp[\theta_1 s_\tau - \tau \psi(\theta_1)]\} .$$

By Lemma 3 and a uniform integrability argument

$$E_0\{(s_\tau - b) \exp[\theta_1(s_\tau - b) - \tau \psi(\theta_1)]\} \rightarrow e^{-\xi} E_0 s_{\tau_+}^2 / 2 E_0 s_{\tau_+} ,$$

so by (25)

$$(26) \quad P_{\theta_0} \{\tau < \infty\} = e^{-\Delta b} [1 - \Delta E_0 s_{\tau_+}^2 / 2 E_0 s_{\tau_+} + o(\Delta)] .$$

It will be notationally convenient to put

$$(27) \quad \beta = E_0 s_{\tau_+}^2 / 2 E_0 s_{\tau_+} .$$

Since the factor  $1 - \beta\Delta + o(\Delta)$  in (26) arises from expanding an exponential function and since  $1 - \beta\Delta + o(\Delta) = e^{-\beta\Delta + o(\Delta)}$  as  $\Delta \rightarrow 0$ , it seems plausible that a better numerical approximation might be provided by taking

$$(28) \quad P_{\theta_0} \{\tau < \infty\} \approx e^{-\Delta(b+\beta)} .$$

It is the content of Theorem 2 below that under slightly stronger assumptions the approximation (28) is valid up to terms which are  $o(\Delta^2)$  rather than the  $o(\Delta)$  provided by (26).

Let  $g(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} G(dx)$  denote the characteristic function of the probability distribution  $G$ . The distribution  $G$  is called strongly non-lattice by Stone (1965) if for each  $\delta > 0$

$$\inf_{|\lambda| > \delta} |1 - g(\lambda)| > 0.$$

Stone shows that for a strongly non-lattice distribution with an exponentially small tail the remainder in the renewal theorem vanishes at an exponential rate. A uniform version of this result leads to

Theorem 2. Assume that  $F$  (the  $P_0$  distribution of  $x_1$ ) is strongly non-lattice. Let  $b \rightarrow \infty$  and  $\theta_0 \uparrow 0$  in such a way that  $\theta_0 b \rightarrow -\xi < 0$ . Then

$$(29) \quad P_{\theta_0} \{\tau < \infty\} = e^{-\Delta b} (1 - \beta \Delta + \beta^2 \Delta^2 / 2 + o(\Delta^2)),$$

where  $\Delta = \theta_1 - \theta_0$  and  $\beta = E_0 s_{\tau_+}^2 / 2 E_0 s_{\tau_+}$ .

Proof. A uniform in  $\theta_1$  version of Stone's (1965) result (see below) applied to the  $P_{\theta_1}$  renewal process defined by  $s_{\tau_+}$  implies that for some  $r > 0$  and all small  $\theta_1$

$$E_{\theta_1} (s_{\tau_+} - b) = E_{\theta_1} s_{\tau_+}^2 / 2 E_{\theta_1} s_{\tau_+} + O(e^{-rb}).$$

The argument preceding (26) shows that

$$E_{\theta_1} (s_{\tau_+} - b)^2 \rightarrow E_0 s_{\tau_+}^3 / 3 E_0 s_{\tau_+}.$$

Substituting these results into (25) and using Lemma 2 completes the proof.

For the most part the required uniformity in Stone's result may be inferred from minor modifications in his argument. The following lemma provides some of the necessary technical results.

Lemma 5. Assume that  $F$  is strongly non-lattice. Then the  $P_\theta$  distributions of  $s_{\tau_+}$  are strongly non-lattice uniformly in small  $\theta \geq 0$ . Also there exists  $r > 0$  such that for each  $\delta > 0$   $E_\theta \exp(zs_{\tau_+})$  is bounded away from 1 in  $0 < \operatorname{Re} z \leq r$ ,  $|z| \geq \delta$ , uniformly in small  $\theta \geq 0$ .

The proof follows by first showing that the  $P_\theta$  distributions of  $x_1$  are uniformly strongly non-lattice and then using the Wiener-Hopf factorization of these distributions (Feller, 1966, p. 570) to show that the  $P_\theta$  distributions of  $s_{\tau_+}$  are uniformly strongly non-lattice. The details have been omitted.

Remark. Although (26) is "obvious," (29) occurs because two terms involving  $E_0 s_{\tau_+}^3 / E_0 s_{\tau_+}$  cancel, and hence is somewhat surprising. It is natural to ask whether this phenomenon might persist, i.e., whether the remainder in (29) is  $-\beta^3 \Delta^3 / 3! + o(\Delta^3)$ . That this is not in general true may be seen from the special case  $F(dx) = \frac{1}{2} e^{-|x|} dx$ , for which  $P_{\theta_0} \{\tau < \infty\}$  may be calculated exactly. See Section 6.

Combining the preceding results with the identities

$$(30) \quad P_{\theta_0} \{s_T > b\} = P_{\theta_0} \{\tau < \infty\} - \int_{\{s_T < a\}} P_{\theta_0} \{\tau < \infty | s_T\} dP_{\theta_0}$$

and

$$(31) \quad P_{\theta_1} \{s_T < a\} = P_{\theta_1} \{\tau^* < \infty\} - \int_{\{s_T > b\}} P_{\theta_1} \{\tau^* < \infty | s_T\} dP_{\theta_1}$$

( $\tau^* = \inf\{n : s_n < a\}$ ) yields approximations for probabilities such as (2). In practice  $|a|$  is frequently 0 (e.g., cumulative sum control charts--see van Dobben de Bruyn, 1968) or of the same order of magnitude as  $b$  (e.g., a symmetric sequential probability ratio test). One example of the possible results is the following.

Theorem 3. Suppose  $a \rightarrow -\infty$ ,  $b \rightarrow \infty$ , and  $\theta_0 \uparrow 0$  in such a way that  $|a|/b \rightarrow c \in (0, \infty)$  and  $\theta_0 b \rightarrow -\xi \leq 0$ . If  $F$  is strongly non-lattice, then

$$(32) \quad P_{\theta_0} \{s_T > b\} = \{1 - \exp[\Delta(a + \alpha)]\} / \{\exp[\Delta(b + \beta)] - \exp[\Delta(a + \alpha)]\} + o(\Delta^2) .$$

Here  $\beta = E_0 s_{\tau_+}^2 / 2 E_0 s_{\tau_+}$  and  $\alpha = E_0 s_{\tau_-}^2 / 2 E_0 s_{\tau_-}$ .

Proof. The first term on the right hand side of (30) is given approximately by Theorem 2. To analyze the second term write

$$\begin{aligned} & \int_{\{s_T < a\}} P_{\theta_0} \{\tau(b) < \infty | s_T\} dP_{\theta_0} \\ &= \left[ \int_{(0, |a|^{1/2})} + \int_{[|a|^{1/2}, \infty)} \right] P_{\theta_0} \{s_T \in a - dx\} P_{\theta_0} \{\tau(b - a + x) < \infty\} . \end{aligned}$$

The integral over  $[|a|^{1/2}, \infty)$  is majorized by

$$\begin{aligned} P_{\theta_0} \{s_T \leq a - |a|^{1/2}\} &\leq P_{\theta_0} \{T > |a|^5\} + P_{\theta_0} \left\{ \sup_{n \leq |a|^5} |x_n| > |a|^{1/2} \right\} \\ &= o(|a|^{-3}) = o(\Delta^3) , \end{aligned}$$

and hence may be neglected. The proof of Theorem 2 shows that uniformly for  $0 \leq x \leq |a|^{\frac{1}{2}}$

$$P_{\theta_0} \{ \tau(b-a+x) < \infty \} = \exp[-\Delta(b-a+x+\beta)] + o(\Delta^2) ,$$

so by Lemma 1

$$\int_{\{s_T < a\}} P_{\theta_0} \{ \tau(b) < \infty | s_T \} dP_{\theta_0} = \exp[-\Delta(b+\beta)] P_{\theta_1} \{ s_T < a \} + o(\Delta^2) .$$

Hence by (30) and Theorem 2

$$P_{\theta_0} \{ s_T > b \} = \exp[-\Delta(b+\beta)] P_{\theta_1} \{ s_T > b \} + o(\Delta^2) .$$

A similar argument starting from (31) yields a second relation for  $P_{\theta_0} \{ s_T > b \}$  and  $P_{\theta_1} \{ s_T < a \}$ , the simultaneous solution of which gives the theorem.

#### 4. $P_{\theta} \{ \tau \leq m \}$ and $P_{\theta} \{ \tau \leq m, s_T > b \}$

The following interpretation of (28) and (32) seems helpful in what follows. The addition of  $\alpha$  and  $\beta$  to  $a$  and  $b$  may be regarded as a continuity correction to be made in replacing the discrete time random walk  $\{s_n\}$  by the continuous time Brownian motion process. The constant  $\Delta$ , which is twice the absolute value of the mean of the approximating Brownian motion process, contains a correction for non-normality of the random walk. If the  $x$ 's are normally distributed with  $\theta = E_{\theta} x_1$ , then  $\theta_0 = -\frac{1}{2} \Delta$  and  $\theta_1 = \frac{1}{2} \Delta$  since no correction for non-normality is necessary. (This interpretation oversimplifies

somewhat, because  $\alpha$  and  $\beta$  are distribution dependent and hence correct for non-normality as well as continuity.)

In approximating the probabilities (3) and (4) the situation becomes more complicated, because a more elaborate correction for non-normality is needed. This correction can be written in a fairly simple form by keeping the preceding interpretation in mind.

For the sake of simplicity only the one-sided stopping rule  $\tau$  is discussed in detail. As in the preceding section,  $b \rightarrow \infty$  and  $\theta \rightarrow 0$  in such a way that  $\theta b = \xi_b \rightarrow \xi$ . An expansion for  $E_\theta \exp(-\lambda\tau/b^2)$  up to terms which are  $o(b^{-1})$  is obtained, and this expansion is formally inverted to give an approximation to  $P_\theta\{\tau \leq b^2 t\}$ . No attempt has been made to justify this inversion, although it is natural to conjecture that the approximation obtained is correct up to terms which vanish faster than  $b^{-1}$ .

Theorem 4. Let  $\gamma = E_0 x_1^3$ , and as before let  $\beta = E_0 s_{\tau_+}^2 / 2 E_0 s_{\tau_+}$ . Assume that  $b \rightarrow \infty$  and  $\theta \rightarrow 0$  in such a way that  $\theta b = \xi_b \rightarrow \xi$ . Let  $h(\lambda, \xi) = (2\lambda + \xi^2)^{1/2} - \xi$ . Then

$$(33) \quad E_\theta \exp(-\lambda\tau/b^2) = \exp[-h(\lambda, \xi_b)(1 + \beta/b)] \\ + (6b)^{-1} \gamma [2\lambda + \xi^2 - \xi^3 / (2\lambda + \xi^2)^{1/2}] \exp[-h(\lambda, \xi)] + o(b^{-1}) .$$

Proof. To simplify the notation assume that  $\xi_b \equiv \xi$ . By Lemma 1, for  $\tilde{\theta} > 0$

$$1 = P_{\tilde{\theta}}\{\tau < \infty\} = E_{\tilde{\theta}} \exp\{(\tilde{\theta} - \theta)s_\tau - \tau[\psi(\tilde{\theta}) - \psi(\theta)]\} .$$

Replacing  $\tilde{\theta}$  by  $\tilde{\theta}/b$  and  $\theta$  by  $\xi/b$  gives

$$\exp[-(\tilde{\theta} - \xi)] = E_{\theta} \exp\{(\tilde{\theta} - \xi)(s_{\tau} - b)/b - \tau[\psi(\tilde{\theta}/b) - \psi(\xi/b)]\}.$$

Using the expansion  $\psi(x) = x^2/2 + \gamma x^3/6 + o(x^4)$  ( $x \rightarrow 0$ ) and setting  $\tilde{\theta} = (2\lambda + \xi^2)^{1/2}$  yields

$$\exp[-h(\lambda, \xi)] = E_{\theta} \exp\{h(\lambda, \xi)(s_{\tau} - b)/b - \lambda\tau/b^2 - \gamma(\tilde{\theta}^3 - \xi^3)\tau/6b^3\} + o(b^{-1}),$$

which by Taylor expansion and application of Lemma 3 and uniform integrability becomes

$$E_{\theta}\{\exp(-\lambda\tau/b^2)[1 - \gamma(\tilde{\theta}^3 - \xi^3)\tau/6b^3][1 + h(\lambda, \xi)(s_{\tau} - b)/b]\} + o(b^{-1})$$

$$= E_{\theta} \exp(-\lambda\tau/b^2) + \beta b^{-1} h(\lambda, \xi) e^{-h(\lambda, \xi)}$$

$$- (6b)^{-1} \gamma(\tilde{\theta}^3 - \xi^3) \int_0^{\infty} t e^{-\lambda t} P_{\theta}\{\tau \in b^2 dt\} + o(b^{-1}).$$

By (18) this last integral converges to  $-\frac{d}{d\lambda} e^{-h(\lambda, \xi)}$ , and hence

$$E_{\theta} \exp(-\lambda\tau/b^2)$$

$$= e^{-h(\lambda, \xi)} (1 - \beta b^{-1} h(\lambda, \xi)) + (6b)^{-1} \gamma[2\lambda + \xi^2 - \xi^3/(2\lambda + \xi^2)^{1/2}] e^{-h(\lambda, \xi)} + o(b^{-1}),$$

which is equivalent to (33).

With the help of (17) and (18) it is an easy matter to invert (33) term by term. However, some additional reflection permits one to write the inversion in a simpler form, which at the same time seems to be more accurate from the point of view of numerical

calculation. In the case  $\xi < 0$ , it seems desirable that if one inverts (33) and then sets  $t = +\infty$ , the resulting approximation should agree with (28). A Taylor series expansion shows that for  $0 = \theta_0 < 0$

$$\exp[-h(\lambda, -\frac{1}{2}\Delta b)(1+\beta/b)] = \exp[-h(\lambda, \xi_b)(1+\beta/b)] \\ + (6b^{-1})\gamma[\xi^2 - \xi^3/(2\lambda + \xi^2)^{1/2}]\exp[-h(\lambda, \xi)] + o(b^{-1}),$$

while for  $\theta = \theta_1 > 0$  a similar result holds with  $h(\lambda, +\frac{1}{2}\Delta b)$ . Thus (33) may be rewritten

$$E_\theta \exp(-\lambda\tau/b^2) = \exp[-h(\lambda, \pm\frac{1}{2}\Delta b)(1+\beta/b)] + \lambda(3b)^{-1}\gamma \exp[-h(\lambda, \xi)] + o(b^{-1}),$$

where  $+\Delta$  is for  $\theta > 0$  and  $-\Delta$  for  $\theta < 0$ . Formal inversion now yields

$$P_\theta\{\tau \leq b^2 t\} \approx G(t; \pm\frac{1}{2}\Delta b, 1+\beta/b) + (3b)^{-1}\gamma G'(t; \xi, 1),$$

which is consistent with (28) for  $t = +\infty$ . A yet simpler approximation, which is consistent with and enlarges upon the interpretation of the first paragraph of this section follows from a further approximation of this last expression by  $G(t + (3b)^{-1}\gamma; \pm\frac{1}{2}\Delta b, 1+\beta/6)$ . Rewriting in terms of  $m = b^2 t$  yields finally

$$(34) \quad P_\theta\{\tau \leq m\} \approx G(m + \frac{1}{3}\gamma b; \pm\frac{1}{2}\Delta, b+\beta).$$

This result has the interpretation that to compute  $P_\theta\{\tau \leq m\}$  approximately one should use the corresponding result for a Brownian motion

with  $b$  replaced by  $b + \beta$  to correct (primarily) for continuity,  $\theta$  replaced by  $\pm \frac{1}{2} \Delta$  to correct for non-normality and  $m$  replaced by  $m + \frac{1}{3} \gamma b$ , also to correct for non-normality.

A similar but more elaborate argument yields an exactly analogous approximation in the case of two barriers. If  $\tilde{T}$  denotes the first time that Brownian motion  $X(t)$  leaves  $(a,b)$ , and  $G_+(t; \xi, a, b) = P_\xi\{\tilde{T} < t, X(\tilde{T}) = b\}$ , then

$$P_\theta\{T \leq m, s_T > b\} \approx G_+(m + \frac{1}{3}\gamma b; \pm \frac{1}{2} \Delta, a + \alpha, b + \beta) .$$

## 5. Numerical Computation of Moments of Ladder Heights

The approximations of the preceding sections involve moments of the ladder heights  $s_{\tau_+}$  and  $s_{\tau_-}$ . In principle the characteristic functions of these random variables are available from random walk theory, and hence the required moments may be calculated. In practice results have been obtained only for special problems. In a related context Woodroffe (1978) suggested a general method based on numerical integration of the characteristic function of the  $x$ 's. In this section Woodroffe's method is adapted to the problems at hand.

By the duality between  $s_{\tau_+}$  and  $s_{\tau_-}$  (Feller, 1966, p. 570), it suffices to consider only  $s_{\tau_+}$ , from which the corresponding results for  $s_{\tau_-}$  may be directly obtained. According to Lai (1976)

$$(35) \quad E_0 s_{\tau_+}^2 / E_0 s_{\tau_+} = \sum_{n=0}^{\infty} \left[ 2^{\frac{1}{2}} \binom{-\frac{1}{2}}{n} (-1)^n - n^{-1} E_0 |s_n| \right] + \frac{1}{3} E_0 x_1^3 ,$$

where  $n^{-1} E |s_n| = 0$  for  $n = 0$ . Hence it is necessary to compute

numerically only one series to use the approximations suggested in the preceding sections.

Let  $f(t) = E_0 \exp(itx_1)$ , and assume that  $\int_{-\infty}^{\infty} |f(t)| dt < \infty$ . (This assumption can be removed by a simple smoothing argument.) Then the distribution function  $F$  has a bounded continuous density  $f_1$  given by the Fourier inversion formula. Let  $f_n$  be the  $n$ -fold convolution of this function, so

$$(36) \quad f_n(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itx} f^n(t) dt .$$

The series in (35) equals

$$(37) \quad \lim_{z \rightarrow 1} \lim_{\lambda \rightarrow 0} \{ 2^{\frac{1}{2}} (1-z)^{-\frac{1}{2}} - \sum_{n=1}^{\infty} n^{-1} z^n E_0 |s_n| \exp(-\lambda |s_n|) \} .$$

The Fourier inversion formula (36), Fubini's theorem, and some calculus gives

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} z^n E_0 |s_n| \exp(-\lambda |s_n|) &= \sum_{n=1}^{\infty} n^{-1} z^n \int_{-\infty}^{\infty} |x| e^{-\lambda |x|} f_n(x) dx \\ &= (2\pi)^{-1} \sum_{n=1}^{\infty} n^{-1} z^n \int_{-\infty}^{\infty} [(\lambda + it)^{-2} + (\lambda - it)^{-2}] f^n(t) dt \\ &= 2\pi^{-1} \int_0^{\infty} (t^2 - \lambda^2)(t^2 + \lambda^2)^{-2} \operatorname{Re} \log(1 - zf(t)) dt . \end{aligned}$$

Observing that  $\int_0^{\infty} (t^2 - \lambda^2)(t^2 + \lambda^2)^{-2} dt = 0$  permits one to rewrite this last expression as

$$\begin{aligned} 2\pi^{-1} \int_0^{\infty} (t^2 - \lambda^2)(t^2 + \lambda^2)^{-2} \operatorname{Re} \log\{[1 - zf(t)]/(1 - z)\} dt \\ = 2\pi^{-1} \int_0^{\infty} (t^2 - \lambda^2)(t^2 + \lambda^2)^{-2} \operatorname{Re} \log\{1 + z[1 - f(t)]/(1 - z)\} dt . \end{aligned}$$

Let  $\gamma = E_0 x_1^3$ . The expansion as  $t \rightarrow 0$

$$(38) \quad 1 + z[1 - f(t)]/(1 - z) = 1 + \frac{1}{2} z (1 - z)^{-1} t^2 [1 + \frac{1}{3} i \gamma t + O(t^2)]$$

is used repeatedly. To evaluate

$$(39) \quad \lim_{\lambda \rightarrow 0} \int_0^\infty t^2 (t^2 + \lambda^2)^{-2} \operatorname{Re} \log \{1 + z[1 - f(t)]/(1 - z)\} dt ,$$

consider the integrals over  $(0, \epsilon)$  and  $(\epsilon, \infty)$  separately. The expansion (38) shows that the integral over  $(0, \epsilon)$  may be made arbitrarily small by taking  $\epsilon$  small enough. The dominated convergence theorem applies to the integral over  $(\epsilon, \infty)$ . Hence, letting  $\lambda \rightarrow 0$  and then  $\epsilon \rightarrow 0$  shows that (39) equals

$$\int_0^\infty t^{-2} \operatorname{Re} \log \{1 + z[1 - f(t)]/(1 - z)\} dt .$$

A similar calculation shows that

$$\lambda^2 \int_0^\infty (t^2 + \lambda^2)^{-2} \operatorname{Re} \log \{1 + z[1 - f(t)]/(1 - z)\} dt \rightarrow 0$$

as  $\lambda \rightarrow 0$ , and hence it suffices to evaluate

$$(40) \quad \lim_{z \rightarrow 1} [2^{\frac{1}{2}} (1 - z)^{-\frac{1}{2}} - 2\pi^{-1} \int_0^\infty t^{-2} \operatorname{Re} \log \{1 + z[1 - f(t)]/(1 - z)\} dt] .$$

Integration by parts shows that

$$\int_0^\infty x^{-2} \log(1 + x^2) dx = \pi ,$$

and hence (40) may be rewritten

$$(41) \quad - \lim_{z \rightarrow 1} 2\pi^{-1} \int_0^\infty t^{-2} \operatorname{Re} \log \left\{ \frac{1 + z[1 - f(t)]/(1 - z)}{1 + t^2/2(1 - z)} \right\} dt .$$

Consider the integrals over  $(0, \epsilon)$  and  $(\epsilon, \infty)$  separately. The expansion (38) shows that for  $t \in (0, \epsilon)$  the argument of the logarithm in (41) equals

$$1 - \left\{ \left[ \frac{1}{2} t^2 + O(t^3/(1-z)) \right] / \left[ 1 + t^2/2(1-z) \right] \right\}.$$

Hence, from the Taylor series for the logarithm one sees that the integral over  $(0, \epsilon)$  is small for small  $\epsilon$ . By the dominated convergence theorem the integral over  $(\epsilon, \infty)$  converges to

$$\int_{\epsilon}^{\infty} t^{-2} \operatorname{Re} \log \{2[1 - f(t)]/t^2\} dt$$

as  $z \rightarrow 1$ . The expansion (38) shows that the integrand is bounded at 0. Hence, letting  $\epsilon \rightarrow 0$  gives the result.

$$(42) \quad E_0 s_{\tau_+}^2 / E_0 s_{\tau_+} = \frac{1}{3} E_0 x_1^3 - 2\pi^{-1} \int_0^{\infty} t^{-1} \operatorname{Re} \log \{2[1 - f(t)]/t^2\} dt,$$

which can be evaluated numerically in many concrete cases.

Remark. The corresponding result for  $E_0 s_{\tau_+}$  is

$$E_0 s_{\tau_+} = 2^{-1/2} \exp\{\pi^{-1} \int_0^{\infty} t^{-1} \operatorname{Im} \log [1 - f(t)] dt\}.$$

## 6. Numerical Examples

For a simple but instructive example let  $F(dx) = \frac{1}{2} e^{-|x|} dx$ .

It is easy to see that  $\theta_1 = -\theta_0$ , so  $\Delta = 2\theta_1$ , and for all  $b \geq 0$  and

$\theta_1 \geq 0$  the  $P_{\theta_1}$ -distribution of  $s_{\tau} - b$  is exponential with mean  $(1 - \frac{1}{2} \Delta)^{-1}$ . Hence, by Lemma 1 applied to  $\theta_0$  and  $\theta_1$

$$P_{\theta_0} \{\tau < \infty\} = e^{-\Delta b} (1 - \frac{1}{2} \Delta) / (1 + \frac{1}{2} \Delta) \quad .$$

Also,  $\beta = E_0 s_{\tau_+}^2 / 2 E_0 s_{\tau_+} = 1$ , so (28) becomes

$$P_{\theta_0} \{\tau < \infty\} \approx e^{-\Delta(b+1)} \quad .$$

The relative error of this approximation is  $e^{-\Delta}(1 + \frac{1}{2} \Delta) / (1 - \frac{1}{2} \Delta) - 1 \approx .01$  for  $\Delta = .5$ . In this example  $P_{\theta_0} \{\tau < \infty\}$  is the stationary probability that the waiting time exceeds  $b$  in a single server queue with Poisson input of intensity  $(1 - \frac{1}{2} \Delta)$  and exponential service with mean  $(1 + \frac{1}{2} \Delta)^{-1}$ . The traffic intensity is  $(1 - \frac{1}{2} \Delta) / (1 + \frac{1}{2} \Delta)$ . The customary diffusion approximation is  $P_{\theta_0} \{\tau < \infty\} \approx e^{-\Delta b}$ , which is valid in "heavy traffic," i.e., for  $\Delta \approx 0$ . For  $\Delta = .5$  the traffic intensity is .6, which is far from heavy traffic, but the approximation (28) is quite good. For this  $\Delta$  the uncorrected diffusion approximation has relative error .67 and doesn't attain 1% accuracy until the traffic intensity is .99.

For a non-trivial and important example suppose that  $F = \Phi$ . The stopping rule  $T$  defines a sequential probability ratio test for deciding whether a normal mean  $\theta$  is positive or negative. Several quantities similar to (4) arise in studying a truncated version of this test. They have been computed by iterative numerical integration by Aroian and Robison (1969). The preceding section suggests that one approximate these quantities by computing their analogues for a Brownian motion process with stopping boundaries at  $b + \beta$  and

$a + \alpha$ . It is easy to use either (35) or (42) to obtain

$$\beta = -\alpha = E_0 s_{\tau_+}^2 / 2E_0 s_{\tau_+} \approx .583.$$

Table 1 gives the results of several exemplary calculations. For comparison the "exact" results of Aroian and Robison (1969) are included as the second entry of each cell. In some cases the unmodified diffusion approximation is given as the third entry. In general, the suggested approximation is good and the diffusion approximation poor.

Even for Brownian motion the exact formulas for the quantities in Table 1 are complicated, and it is worth noting that simple approximations suffice. For example, if  $X(t)$  is Brownian motion with drift  $\theta$  and  $\tilde{T} = \inf\{t: X(t) \notin (a, b)\}$ , then (Ito and McKean, 1965, p. 31)

$$P_0\{\tilde{T} > t, X(t) \in dy\} = 2 \sum_{n=1}^{\infty} \exp[-n^2 \pi^2 t / 2(b-a)^2] \sin \frac{n\pi|a|}{b-a} \sin \frac{n\pi(y-a)}{b-a} dy / (b-a).$$

For large  $t$ , all terms in this series are negligible compared to the first, and hence by Lemma 1 as  $t \rightarrow \infty$

$$\begin{aligned} P_{\theta}\{\tilde{T} > t\} &= \int_a^b P_0\{\tilde{T} > t, X(t) \in dy\} \exp(\theta y - \theta^2 t / 2) \\ &\sim \exp\left\{\theta a - \frac{1}{2} \theta^2 t \left[1 + \pi^2 / \theta^2 (b-a)^2\right]\right\} [e^{\theta(b-a)} + 1] \sin[\pi|a|/(b-a)] \frac{2\pi}{\pi^2 + \theta^2 (b-a)^2}. \end{aligned}$$

The next example illustrates the accuracy of the corrections for non-normality. Let  $X_1, X_2, \dots$ , be independent and exponentially distributed with expectation  $\lambda^{-1}$ . Let  $\lambda_0 = 1$ ,  $\lambda_1 = 1.5$  and  $\lambda^* = (\lambda_1 - \lambda_0) / \log(\lambda_1 / \lambda_0) \approx 1.233$ . Let  $s_n = n - \lambda^* \sum_{k=1}^n X_k$  and

TABLE 1  
STOPPING RULE T WITH  $a = -b$ ; NORMAL VARIABLES

Parameters	$P_{\theta}\{T > m_2\}$	$P_0\{T > m_2\}$	$P_{-\theta}\{T \leq m_1, s_T > b\}$ $+P_{-\theta}\{T > m_1, s_{m_1} > 0\}$	$E_0(T \wedge m_1)$
$b=5.889, \theta=.25$ $m_1=66, m_2=44$	.11, .11, .082	.35, .35, .27	.044, .044	35.7, 35.9
$b=2.944, \theta=.5$ $m_1 = m_2 = 11$	.14, .14, .082	.43, .42, .27	.055, .056	8.1, 8.3

$T$  = first  $n$  such that  $s_n \notin [a, b]$ . The stopping rule  $T$  is that of a sequential probability ratio test for testing  $\lambda = \lambda_0$  against  $\lambda = \lambda_1$ . For the specific choice  $-a = 4.419, b = 5.419$  the Wald approximations to the error probabilities  $P_{\lambda_0}\{s_T > b\}$  and  $P_{\lambda_1}\{s_T < a\}$  are both 0.1. (Here  $P_{\lambda}$  denotes probability when the exponential parameter is  $\lambda$  and corresponds to but is not identical with the  $P_{\theta}$  of earlier sections.) For various values of  $\lambda_0, \lambda_1, a$  and  $b$  this test, usually truncated, is widely used in quality control and reliability studies. Numerical computations of its exact error probabilities as well as various probabilities of the form (4) have been carried out by Epstein, Patterson, and Qualls (1963). These latter probabilities are of interest in studying truncated versions of the test and in the theory of confidence intervals following sequential tests suggested by Siegmund (1978).

The random variable  $-s_{T-}$  has an exponential distribution with mean  $\lambda^*/\lambda$ . It follows easily that  $\alpha = -1$  and  $\beta = \frac{1}{3}$ . The approximation (32) gives  $P_{\lambda_0}\{s_T > 5.419\} \approx .087$  and  $P_{\lambda_1}\{s_T < -4.419\} \approx .101$ , which are both exact to three decimal places. Table 2 compares the approximate value of (4) recommended in Section 4 to exact values computed by Epstein, Patterson, and Qualls (1963).

TABLE 2  
SEQUENTIAL TEST OF  $\lambda = \lambda_0 (= 1)$  AGAINST  $\lambda = \lambda_1 (= 1.5)$   
WITH  $a = -4.419$  AND  $b = 5.419$

	$\lambda_0$	$\lambda^*$	$\lambda_1$
$P_{\lambda}\{T \leq 30, s_T > b\} \approx$	.060	.259	.620
$=$	.060	.261	.624
$P_{\lambda}\{T \leq 30, s_T < -a\} \approx$	.746	.346	.083
$=$	.751	.351	.084

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